It's Not Hyperbole: A Transforming Proof

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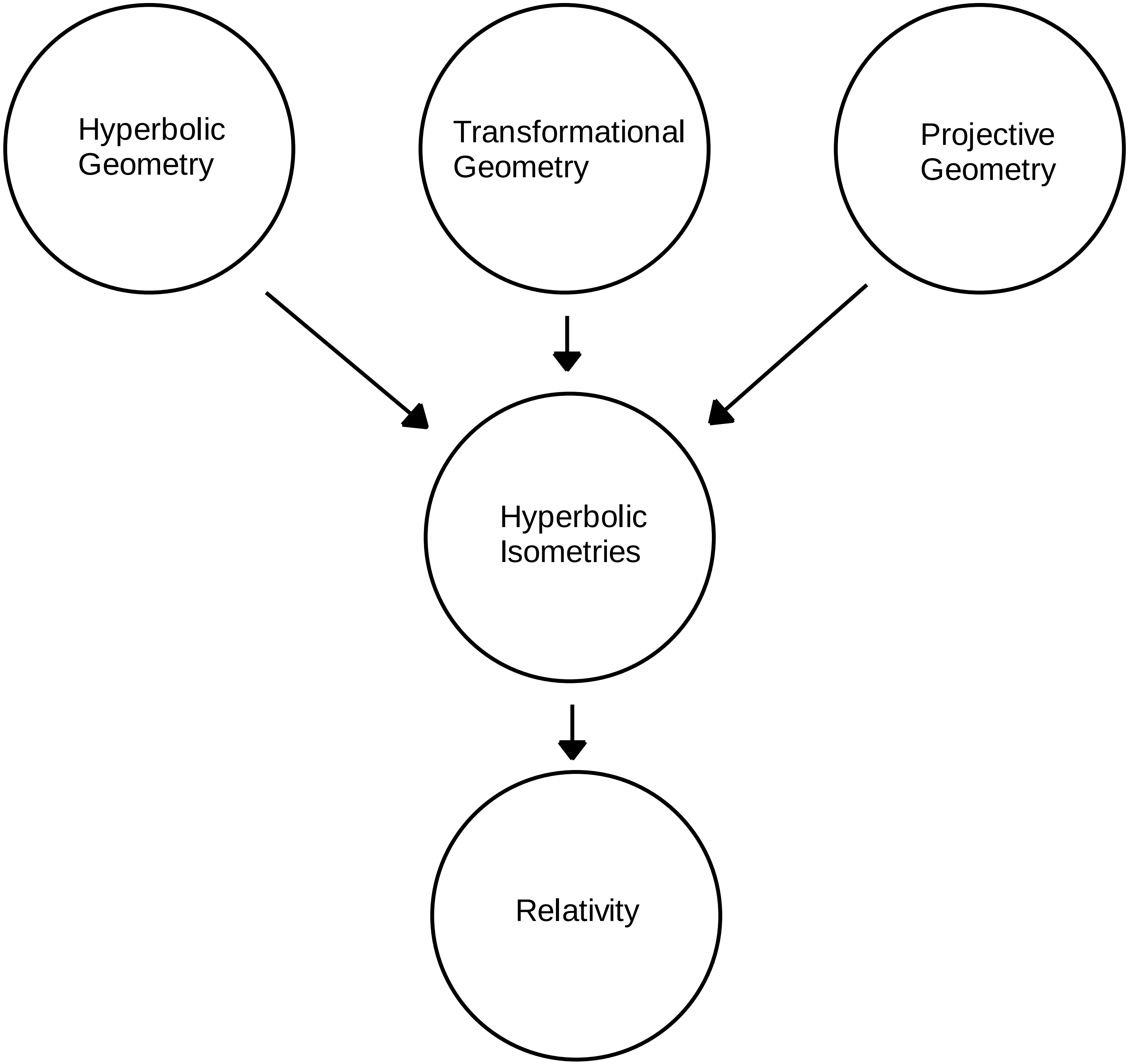

Hyperbolic Geometry
Transformational Geometry
Projective Geometry
Hyperbolic Isometries
Relativity

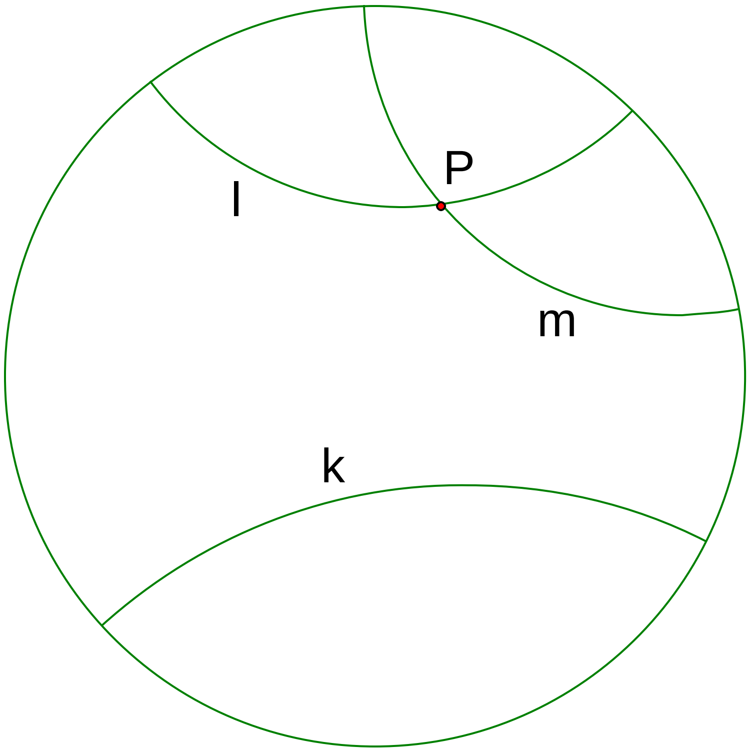
Topics in Geometry Course

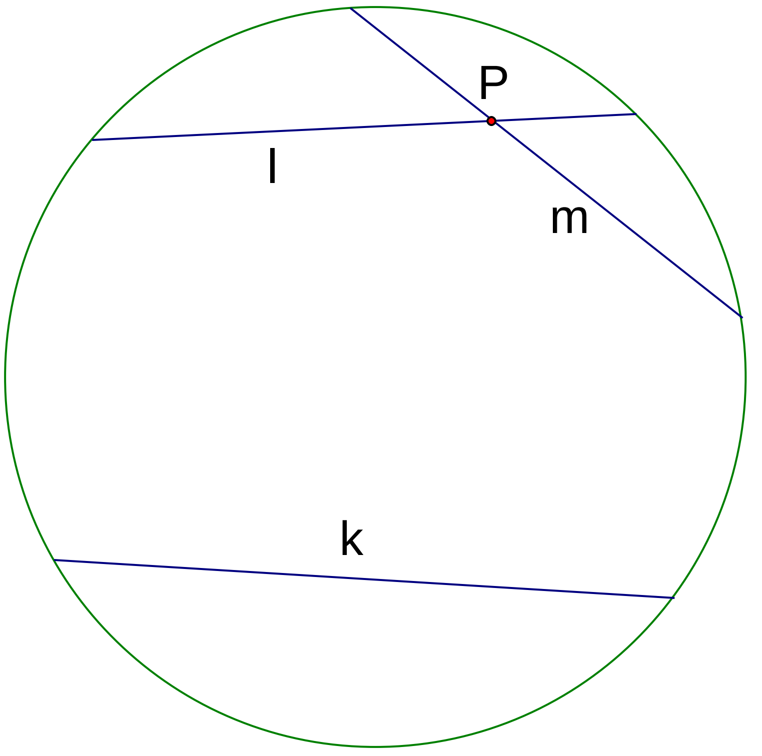
- Euclidean Geometry, Axiomatics
- Non-Euclidean Geometry
- Transformational Geometry, Symmetry
- Survey of Projective Geometry

Models of Hyperbolic Geometry

- Poincaré Model

Klein Model





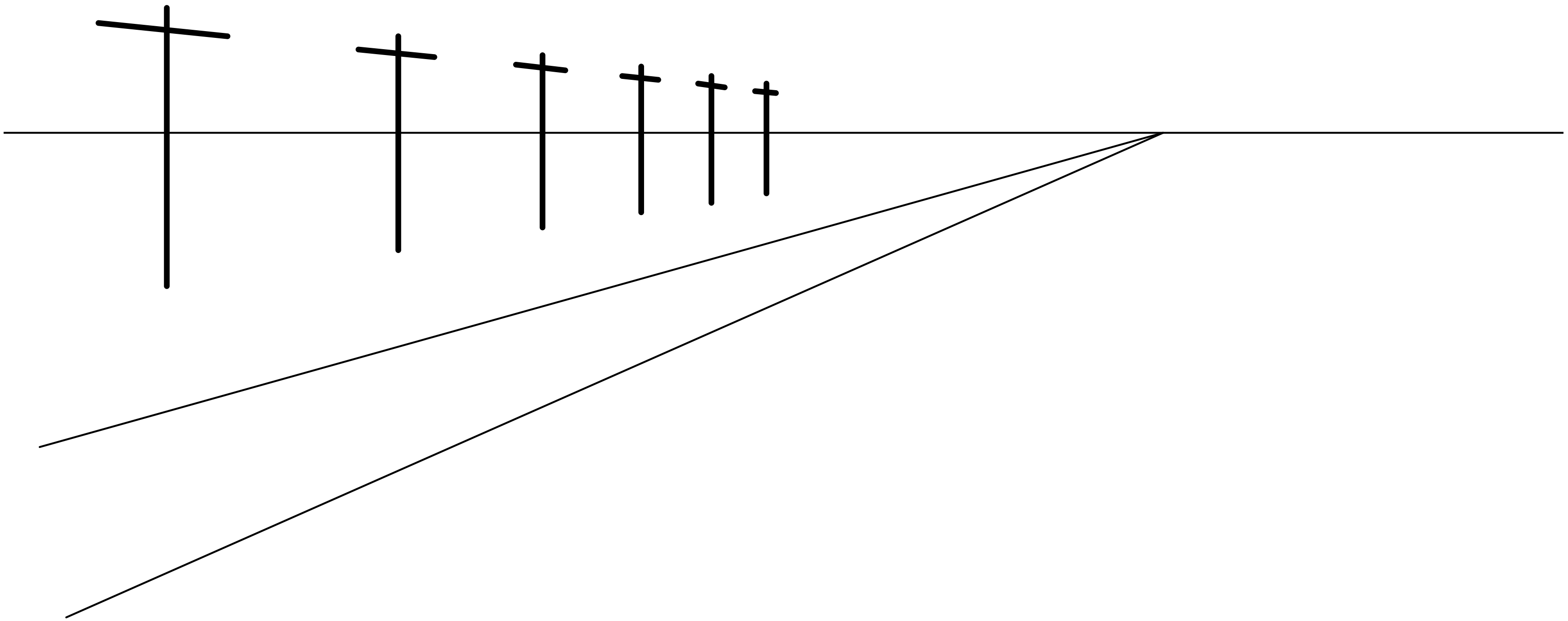
Plane Transformations

- 2x2 matrices fix origin.
- Use plane z = 1 in $\mathbf{R}^3$. Points are $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ and transformations are $\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$.

Spherical Isometries

- A 3x3 matrix M is a spherical isometry iff it is orthogonal. That is, $M^T = M^{-1}$ or its columns are mutually orthogonal and have length 1:
- For columns M_1, M_2 and M_3,
  $M_j \cdot M_k = 0$ if $j \neq k$ and $M_j \cdot M_j = 1$.

Projective Geometry



Analytic Projective Geometry Homogeneous Coordinates

- Points: 3x1 column vectors.
- Lines: 1x3 vectors. P is on k iff $k \cdot P = 0$.
- Conics: 3x3 symmetric, nonsingular matrices. P is on C iff $P^T \cdot C \cdot P = 0$.
- Representations of points (lines, conics) are equivalent if they differ by a nonzero scalar.

- Unit circle has equation $x^2 + y^2 - z^2 = 0$ or

$$[x\ y\ z]\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = 0.$$

- Note: Not quite dot product of the point with itself.

Collineations (Projective Transformations)

- A collineation is a nonsingular 3x3 matrix.
- M leaves point P fixed iff $MP = \lambda P$.

Collineations
(Projective Transformations)

- A collineation is a nonsingular 3x3 matrix.
- M leaves point P fixed iff $MP = \lambda P$.
- M takes line k to kM^{-1}
  because $(kM^{-1})(MP) = kP$ so the image of P is on image of k iff P is on k.
- M leaves line k stable iff $kM^{-1} = \lambda k$.

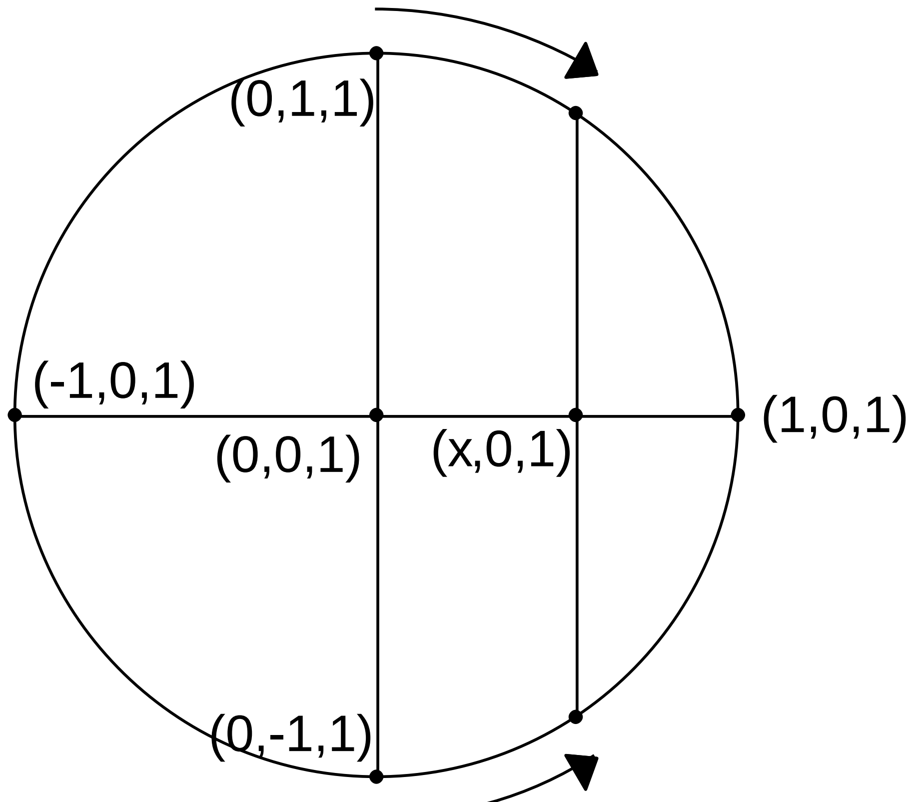
Collineations (Projective Transformations)

- A collineation is nonsingular 3x3 matrix.
- M leaves conic C stable iff
  $(M^{-1})^T C (M^{-1}) = \lambda C$.

- Lemma. The set of collineations leaving a conic stable forms a group under composition.

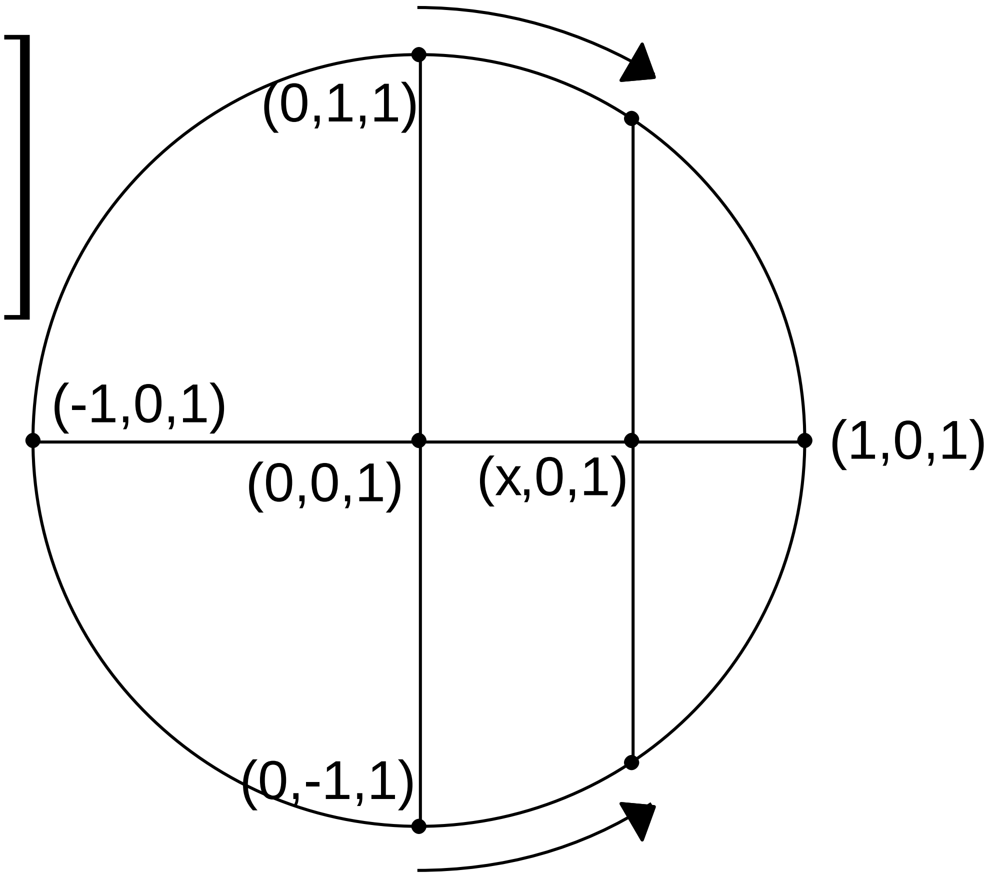
Hyperbolic Isometries for Klein Model

- Definition. A collineation is a *hyperbolic isometry* iff it leaves the unit circle stable.



Hyperbolic “Translation”

- $$\begin{bmatrix} 1 & 0 & x \\ 0 & \sqrt{1-x^2} & 0 \\ x & 0 & 1 \end{bmatrix}$$



- Definition. The *hyperbolic inner product* of two column vectors P = $\begin{bmatrix} s \\ t \\ u \end{bmatrix}$ and Q = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is P $\cdot_h$ Q = $sx + ty - uz$.
- Note: This is not an inner product for linear algebra.

- **Theorem**. A collineation with columns M_1 , M_2 and M_3 is a hyperbolic isometry iff $M_j \cdot_h M_k = 0$ if $j \neq k$ and $M_1 \cdot_h M_1 = M_2 \cdot_h M_2 = -(M_3 \cdot_h M_3)$.

- Compare: Spherical isometry iff $M_j \cdot M_k = 0$ if $j \neq k$ and $M_j \cdot M_j = 1$.

- **Theorem**. A collineation with columns M_1 , M_2 and M_3 is a hyperbolic isometry iff $M_j \cdot_h M_k = 0$ if $j \neq k$ and $M_1 \cdot_h M_1 = M_2 \cdot_h M_2 = -(M_3 \cdot_h M_3)$.

- $$\begin{bmatrix} 1 & 0 & x \\ 0 & \sqrt{1 - x^2} & 0 \\ x & 0 & 1 \end{bmatrix}$$

- Proof. Let M be a collineation.

For $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ to be stable we must have $(M^{-1})^T C (M^{-1}) = \lambda C$. Let the columns of M^{-1} be X, Y and Z. Then $(M^{-1})^T C (M^{-1}) =$

- Then $(M^{-1})^T C (M^{-1})$ =

$$\begin{bmatrix} X^T \\ Y^T \\ Z^T \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} X & Y & Z \end{bmatrix} = \begin{bmatrix} X \cdot_h X & X \cdot_h Y & X \cdot_h Z \\ Y \cdot_h X & Y \cdot_h Y & Y \cdot_h Z \\ Z \cdot_h X & Z \cdot_h Y & Z \cdot_h Z \end{bmatrix}.$$

- Then $(M^{-1})^T C (M^{-1}) = \lambda C$ iff

$$\begin{bmatrix} X \cdot_h X & X \cdot_h Y & X \cdot_h Z \\ Y \cdot_h X & Y \cdot_h Y & Y \cdot_h Z \\ Z \cdot_h X & Z \cdot_h Y & Z \cdot_h Z \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}.$$

- That is, the hyperbolic inner products of different columns are all 0 and $X \cdot_h X = Y \cdot_h Y = -Z \cdot_h Z$.
- Thus M leaves C stable iff M^{-1} is a hyperbolic isometry.
- The collineations leaving C stable are a group, so M leaves C stable iff M is a hyperbolic isometry. Q. E. D.

Theory of Special Relativity and Hyperbolic Isometries

- Additivity of velocities corresponds to compositions of hyperbolic translations.

- Minkowski geometry is a subgeometry of four-dimensional projective geometry. Lorentz transformations leave invariant the quantity $\Delta x^2 + \Delta y^2 + \Delta z^2 - \Delta t^2$.